Article

Complementary Time-Dependent Coordinate Transformation

Alexandru C. V. Ceapa*

ABSTRACT

Abstract coordinate systems at abolsute rest are discussed. Time-dependent coordinate transformations that are complementary to those already known as spatial translations and rotations are described here. Then, it is shown that standard Lorentz transformation is a complementary time-dependent coordinate transformation.

Key Words: complementary, time-dependent, coordinate transformation, abstract coordinates, absolute rest, inertial coordinate system.

6. ABSTRACT COORDINATE SYSTEMS AT ABSOLUTE REST

We give evidence for abstract coordinate systems at absolute rest associated to inertial coordinate systems called “at rest” [1] and abstract coordinate systems at absolute rest that professional inertial observers (professionals) associate to their own inertial coordinate systems. Professionals are common inertial observers (uselessly assumed till now to be innocent) *a priori* trained to investigate graphically both seen and unseen relative motions.

6.1. Abstract Coordinate Systems at Absolute Rest Associated to Coordinate Systems “at Rest”

Consider the diagrams in Fig. 1, with arrows temporarily ignored. In the first diagram, the coordinate system k is moving with constant speed v along the positive common x', x axis relative to a hypothetical coordinate system at absolute rest K. In the second diagram, k moves with the same speed relative to K_1, but k and K_1 are carried by an inertial space of speed w. The coordinate system k coincided with both K and K_1 at $t = 0$. P(x') is a fixed point in k. At time t the second diagram differs from the first one in that everything is shifted right by a distance wt. The Galileo transformation

$$x' = x - vt \tag{1}$$

is predicted by both diagrams. This fact ‘entitled’ observers to name their inertial coordinate systems “at rest”, and to treat them as coordinate systems at absolute rest.

Consider further the same diagrams with arrows drawn. They stand for physical signals tracing radius vectors of geometrical points moving with respect to observer. Among all possible physical signals, *we here, and in subsequent diagrams, select light signals. We do it to pregnantly emphasize the deep connection of our results with Einstein’s special relativity theory. The generality of all the obtained formulas is assured by changing c to v within them, where v stands for the speed of whichever signal.*

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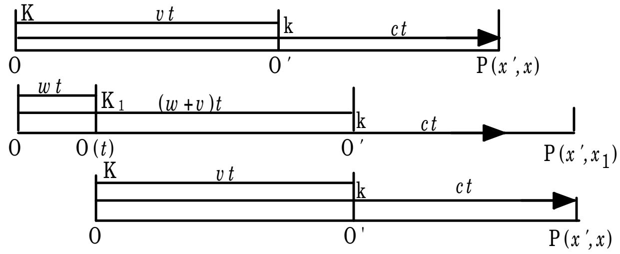

Figure 1.

Let the arrows on Fig. 1 stand for the light signal tracing the radius vector of P(x'). At time $t=0$, this signal and the origin of k leave the origin of K, K_1, respectively, moving along the x', x axes with speeds c, v. At time t, they reach, respectively, P and O' in the first diagram, and we get Eq. (1) with $x = ct$. Also at time t, the path of the signal in the second diagram is ct, but both the origin of K_1 and P are shifted right to O(t) and P(x', x_1) for the distance wt. At time $t_1 = t + wt/c$ the light signal will reach P(x', x_1), but in the time wt/c, P(x') moved from P(x', x_1) to P(x', x_2) in the diagram of Fig. 2.

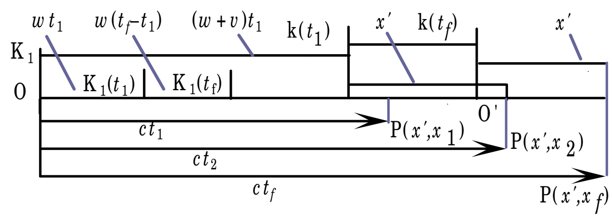

Figure 2.

At time $t_2 = t_1 + (w+v)wt/c^2$, the light signal will reach P(x', x_2), while k, K_1 and P(x') moved further to right by $(w+v)w^2t/c^2$, and $(w+v)^2wt/c^2$, respectively. So that, the time t_f, at which k and K_1 will reach positions denoted respectively by k(t_f) and K_1(t_f), and the light signal P(x') at P(x', x_f), tracing its radius vector relative to O, is given by

$$t_f = t + wt/c + (w+v)wt\big/c^2 + (w+v)^2wt\big/c^3 + \ldots$$
$$= t + wt/c + (w+v)wt\big/c(c - w - v)$$

where the sum of an infinite geometric series of common ratio $(w+v)/c < 1$ was taken into account. At time t_f, the radius vectors of P(x') and of the origin of k, respectively, are located at

$$x_f = ct_f = ct + wt + (w+v)wt\big/(c - w - v)$$

and

$$x_{O'} = (v+w)t_f = vt + wt + (w+v)wt\big/(c - w - v)$$

So $x_f - x_{O'}$ reduces to Eq. (1) by removing the line segments OO(t)$=wt$ and $P(x', x_1)P(x', x_f) = (w+v)wt\big/(c-w-v)$ covered by the light signal and the origin of k, in accord with the second diagram in Fig. 1. The third diagram in Fig. 1 follows. By that the radius vector of the geometrical point P(x') is traced by the signal in time t, this diagram associates the 'abstract' coordinate system at absolute rest K to the observer's inertial coordinate system K_1.

Therefore, the very graphical and mathematical description of the uniform rectilinear motion of any object relative to an inertial observer is done with respect to the coordinate system at absolute rest associated to his inertial coordinate system. The 'relative' speed appears to be an absolute quantity (that is one defined with respect to a coordinate system at absolute rest).

6.*1.1.* *The 'Relativistic' Law of Addition of Parallel Speeds*

Consider now the diagrams in Fig. 3. The coordinate system at absolute rest K is that above associated to K_1. The k_A, k and K coincide at $t_0 = 0$. Just at $t_0 = 0$, k_A, k and a light signal, tracing the radius vector of P fixed in k, leave the origin O of K. They move uniformly along the common x', x'', x axis with speeds v, w and c, respectively. At time t, their origins and the tip of the signal reach, respectively, the points $O'_A(vt)$, $O'(wt)$ and $Q(ct)$ in the upper diagram. By diagrams like the last two in Fig. 1, with K_1, K changed to k_A, K_A, we turn the motion of k relative to k_A to one relative to the coordinate system at absolute rest K_A associated to the inertial k_A. To this end, the light signal and the origin of k must continue their motion an additional time vt/c, until reaching P and $O'[w(t+vt/c)]$, respectively.

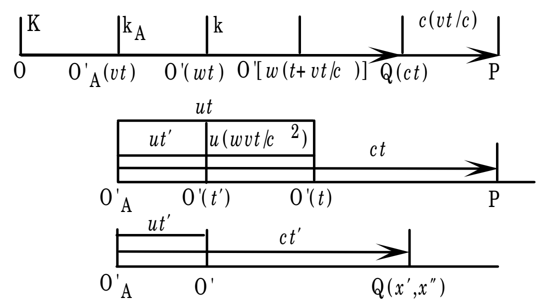

Figure 3.

Since O'_AP was traveled by the signal in time t, the bottom diagram in Fig. 1 is regained as the second one in Fig. 3, where $O'(t)$, $O'(t')$ stand for the origin of k relative to O'_A at times t, t', respectively. For a speed u of k relative to K_A, this diagram predicts the relationship $ut' = (w-v)t$ at the time $t' = t - wvt/c^2$ and, by simplification, the equation

$$u = (w-v)/(1-wv/c^2) \qquad (2)$$

The speeds u, v, w in Eq. (2) are absolute quantities (as defined in Sect. 6.1). u defines the speed of motion of k with respect to the fixed point O'_A. All happens as if the origin of k_A was at rest at O'_A in the time t, and that of k moved at $O'(wt)$ with speed u in the time t'. u is a true speed: u, and not w-v, serves to calculate the kinetic energy of a body at rest in k, releasable with respect to k_A. It is this reason for which u given by Eq. (3) is used in the relativistic kinematics.

Therefore, for c changed to v, the 'relativistic' law of addition of parallel speeds given by Eq. (2) is specific to any theory in which the radius vectors are traced by physical signals.

6.*1.2.* *Complementary Time-Dependent Coordinate Transformation for Geometrical Points Located on the Observer's Direction of Motion: Particular Form*

Observe that the first diagram in Fig. 3 predicts for Q the set of equivalent equations

$$x' = x - vt,\ t' = t - vx/c^2 \quad (3)$$

Also observe that, for a geometrical point -the origin O' of k- moving with the absolute speed *w*, the additional equation $x = wt$ assures the independence of Eqs. (3). So Eqs. (3) define a coordinate transformation. According to Sec. 2, this is a complementary time-dependent coordinate transformation connecting coordinates -defined with respect to the coordinate systems at absolute rest K and K_A- of geometrical points located on the observer's direction of motion. Since Eqs. (3) and the equations

$$x'' = x - wt,\ t'' = t - wx/c^2,$$

also predicted by the first diagram, give rise to the equations

$$x'' = x' - ut',\ t'' = t' - ux'/c^2$$

predicted by the last diagram, the coordinate transformations of type (3) form a group.

6.2. Abstract Coordinate Systems at Absolute Rest Associated to Coordinate Systems of Inertial Observers

A professional at rest with respect to the origin of k in Fig. 1, can always associate
coordinate systems at absolute rest (K, Ξ) to, respectively, the inertial coordinate systems K_1 and k by reflecting at point P(x') fixed in k, the light signal tracing its radius vector, as depicted in the diagrams in Fig. 4. The first because, as a point of space, hence at absolute rest, the origin O'_o of the signal defines the origin O of K. The last in view of the equations

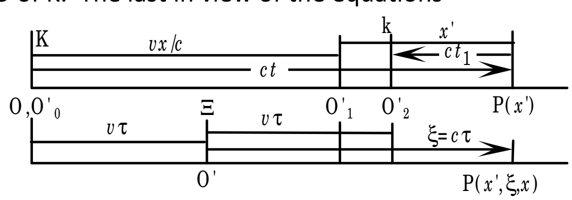

Figure 4.

$$x' = x - vt,\ x' = vt_1 + ct_1 \quad (4)$$

having as solutions

$$t = x'/(c - v),\ t_1 = x'/(c-v). \quad (5)$$

Thus defining

$$\tau = (t + t_1)/2,\ \xi = c\tau, \quad (6)$$

he obtaines equations $\tau = \beta^2 x'/c$, $\beta=\beta^2 x'$, and implicitly

$$O'_oO'_2/2 = c(t-t_1)/2 = v\tau,\ x = ct = \xi + v\tau. \quad (7)$$

Since x' is the abscissa of a point P fixed in k, it is constant. The quantities τ and ξ are also constants. Therefore, the point O' of abscissa $v\tau$ is a fixed point in K. Since ξ gives the position of P relative to O', the last of Eqs. (7) defines O' as the origin of a coordinate system at absolute rest Ξ

associated to the inertial coordinate system k. As depicted in the second diagram in Fig. 4, Ξ is parallel to k and K, having in common the x', ξ, x axis. The radius vector of P relative to Ξ, ξ is traced by a light signal in the time τ of Ξ. By Eqs. (5), (6) and (1), and adding equations $\eta = y$, $\zeta = z$, he gets

$$\xi = \beta^2(x - vt),\ \eta = \beta y,\ \zeta = \beta z,\ \tau = \beta^2(t - vx / c^2), \qquad (8)$$

where $\beta = 1/\sqrt{1 - v^2/c^2}$, which connect coordinates of P relative to the coordinate systems at absolute rest Ξ, K.

7. GRAPHICAL ADDITION OF TRAVEL TIMES AS SCALAR QUANTITIES

The parallelogram rule of addition of two vectors making the angle α with each other gives by the extended Pythagorean theorem

$$t=(t_1^2+t_2^2+2t_1t_2\cos\alpha)^{1/2} \qquad (9)$$

as the formula for adding travel times elapsed by light along such vectors. Eq. (9) conflicts with the scalar feature of time, and must be abolished.

To this end we first consider a sequence of collinear line segments OA_1, A_1A_2,…, $A_{n-1}A_n$ in empty space, and denote

$$OA_n = OA_1+A_1A_2+\ldots+A_{n-1}A_n \qquad (10)$$

Because the time in which a light signal travels any line segment is the difference between the times indicated by synchronous clocks located at its endpoints at the arrival of that signal [in our case t(O), t (A_1), … , t (A_n)], we always have

$$t\ (OA_n) = t\ (OA_1)+ t\ (A_1A_2)+\ldots+ t\ (A_{n-1}A_n) \qquad (11)$$

with $t\ (OA_n) = t(A_n) - t(O) = OA_n/c$, $t(OA_1) = t(A_1) - t(O) = OA_1/c$, $t(A_1A_2) = t(A_2) - t(A_1) = A_1A_2/c$, …, $t(A_{n-1}A_n) = t(A_n) - t(A_{n-1}) = A_{n-1}A_n/c$.

When obtained dividing a geometrical equation like (10) by the speed of a physical signal (in particular that of light), Eq. (11) defines what we here call *graphical addition of travel times as scalar quantities*. The derivation of Eq. (11) from Eq. (10) is basic in a theory manipulating physical signals, as the special relativity theory is.

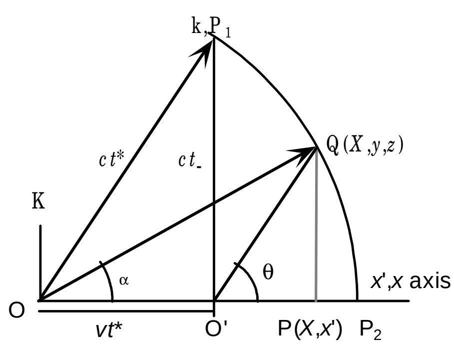

Figure 5.

The choice of collinear light signals in [1] has hidden the case of the collinear line segments which depend on travel times of non-collinear light signals, like those tracing the radius vectors OQ, O'Q in the diagram in Fig. 5, with k and K in Sec. 5 (Sect. 1.1). The collinear line segments OO', O'P and OP are covered respectively with speeds v, $c\cos\alpha$ and $c\cos\theta$ by the origin of k and the projections onto the common x', x axis of the tips of the light signals tracing OQ, O'Q. Therefore they depend on the travel times t^* and O'Q$/c$. Evidently, this prevents us from getting a time equation like (11) by simply dividing equation OO'+O'P=OP by c. In order to get such an equation, we need to express OP, OO' and O'P in terms of the travel time of one and the same light signal. This means that we need to relate them geometrically to the path of such a signal (O'P_1 in Fig. 5). We name *time-axis* the direction orthogonal to $\mathbf{v}$. By applying the Pythagorean theorem to the right triangle OP_1O', we have

$$t^* = \beta t_- \qquad (12)$$

Laying O'O and OP on the time-axis is straightforward. Similarly expressing O'P as the path of a light signal fails, in which case we must identify different geometry avoiding the dependence of O'P on O'Q/c.

Consider the diagram in Fig. 6, also with k and K in Sec. 5 (Sect. 1). Q, Q_1, and P(X), P(βX) as their projections, are fixed points relative to k. At time $t=0$, the origin of k and the light signal traveling to P(X) leave the origin O of the coordinate system at absolute rest K. At time T [(r/c)·cosα], they reach, respectively, O'_o and P(X). We lay the bottom diagram in Fig. 4 at O'_o on the time-axis $O'_oP'_1$ which means that we refer the motion of k to the coordinate system at absolute rest Ξ. For the reason leading to (12), from the right triangle $OP'_1O'_o$ we have

$$T = \beta t,\ X = cT = \beta ct = \beta x,\ OO'_o = vT = v\beta t. \qquad (13)$$

By Eqs. (4), (13) we further determine ξ and τ in terms of X and T. We get

$$\xi = \beta(X - vT),\ \eta = y,\ \zeta = z,\ \tau = \beta(T - vX/c^2). \qquad (14)$$

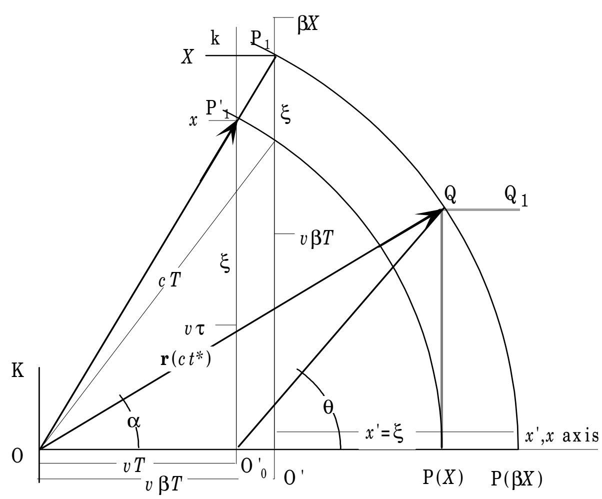

Figure 6.

Thus, by passing from Q to the geometrical point Q_1, we get rid of the dependence of the abscissa of P on the time O'_oQ/c. The abscissa of Q_1 relative to K is β times that of Q. It is ξ with respect to both k and Ξ: Since ξ is traveled by a light signal in time τ, the abscissa of Q_1 relative to k is also traveled in time τ.

Therefore, a time equation like that given by (11) follows immediately along the x', x axis, dividing by c the equation OO'+O'P(βX)=OP(βX). So we passed from Eq. (9) to one of type (11), adding Newtonian travel times as scalar quantities.

8. COMPLEMENTARY TIME-DEPENDING COORDINATE TRANSFORMATIONS FOR GEOMETRICAL POINTS OFF THE OBSERVER'S DIRECTION OF MOTION: GENERAL FORM

As a straightforward consequence of the graphical addition of travel times as scalar quantities (developed in Sec. 7), Eqs. (14) give, for any geometrical point P($x'.x$) and physical signal of speed υ, the set of equations

$$x' = \beta(x - \upsilon t),\ y' = y,\ z' = z,\ t = \beta(t - \upsilon x/v^2), \tag{15}$$

with $\beta = (1-\upsilon^2/v^2)^{-1/2}$.

For Eqs. (15) to express a coordinate transformation, we must brake the equivalence of the first and the last of them. To this end, consider the Q's (implicitly their projections P) in Fig. 6 to move relative to the coordinate system k, which is also in uniform translatory motion relative to K. Identifying P with the origin of the coordinate system k, we are in the case pointed out in the last paragraph of Sec. 6 (Sect. 1.2). So, we pass from a description of the motion of Q relative to the inertial coordinate system k to one with respect to a coordinate system at absolute rest K_A associated to k just as it was associated to k_A in Sec. 6 (Sect. 1.1). By a diagram analogous to the last one in Fig. 3 and by the additional equation $x = wt$ analogous to that associated to Eqs. (3), we break the equivalence of the first and the fourth of Eqs. (15).

Thus, with the additional equation $x = wt$, Eqs. (15) give the general form of the 'complementary time-dependent coordinate transformations' due to the tracing of the radius vectors of moving geometrical points off the common x', x axis with physical signals. The term βx in Eqs. (15) is the Cartesian coordinate of a geometrical point associated to P($x'.x$) in consequence of the graphical addition of travel times as scalar quantities, βt is a Newtonian time -that in which the physical signal travels the coordinate βx -, while $\beta\upsilon t$ is the Cartesian coordinate of another geometrical point -the origin of the inertial coordinate system.

9. THE STANDARD LORENTZ TRANSFORMATION AS A COMPLEMENTARY TIME-DEPENDENT COORDINATE TRANSFORMATION

Tracing the radius vectors of moving geometrical points with light signals (as depicted in the diagrams in Figs. 5, 6), Eqs. (15), written for *v=c*, give the standard Lorentz transformation as a 'complementary time-dependent coordinate transformation'. As a 'complementary time-dependent coordinate transformation' connects finite Cartesian coordinates and Newtonian times, the standard Lorentz transformation connects evidently finite Cartesian coordinates1 and Newtonian times (βx,

[1] Our derivation of the Lorentz transformation as a 'complementary time-dependent coordinate transformation' deny the claim in [6] that the Lorentz transformation would always connect "infinitesimals instead of finite" coordinates. For an observer attached to the origin of S' (the equivalent of our k) in the diagram in [6], and tracing radius vectors by light signals, there is neither the claimed paradox nor the need that the Lorentz transformation to connect infinitesimals.

βvt and βt , respectively) neither spatial and time intervals nor a coordinate (x) and a fictitious time (t) multiplied by a factor (β) of mysterious origin and physical meaning.

The derivation of the Lorentz transformation as a complementary time-dependent coordinate transformation validates our working hypotheses (see Sec. 3).

10. OPERATIONAL DERIVATION OF THE VECTOR LORENTZ TRANSFORMATION

Consider the diagram in Fig. 7. The coordinate system k moves rectilinearly with constant speed v relative to the coordinate system at absolute rest K along the direction $\hat{\mathbf{v}} = \mathbf{v} / v$.

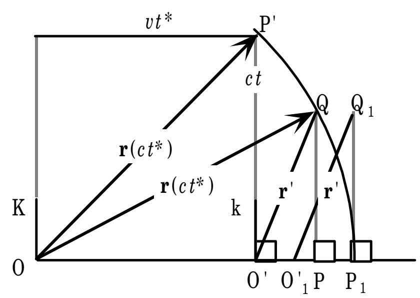

Figure 7.

A light signal traveling OP in time T is used, just like in Sec. 7 above (O'P' playing the role of time-axis) to remove the dependence of OP and O'P on t^* and O'Q/c, respectively. So we pass from Q and O' to Q_1 and O'_1 with $OP_1=\beta OP$ and $OO'_1=\beta OO'$. From the right triangles $O'_1Q_1P_1$ and OQP we have $\mathbf{r}'=\mathbf{Q_1P_1}+\mathbf{O'_1P_1}$ with $\mathbf{Q_1P_1}=\mathbf{r}-(\mathbf{r}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$ and $\mathbf{O'_1P_1}=\mathbf{OP_1}-\mathbf{OO'_1}=\beta[(\mathbf{r}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}-\mathbf{v}T]$, that by noting, $t'=\mathbf{r}'\cdot\hat{\mathbf{v}}/c$ and $T=\mathbf{r}\cdot\hat{\mathbf{v}}/c$, provides the vector Lorentz transformation as

$$\mathbf{r}' = \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} + \beta[(\mathbf{r}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} - \mathbf{v}T],\ t' = \beta[T - \mathbf{r}\cdot\mathbf{v}/c^2]. \tag{16}$$

From a diagram analogous to that in Fig. 7, describing the rectilinear motion of constant speed $\mathbf{w}$ of a coordinate system k relative to the coordinate system at absolute rest K, we obtain analogously the vector Lorentz transformation

$$\mathbf{r}'' = \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{w}})\hat{\mathbf{w}} + \gamma[(\mathbf{r}\cdot\hat{\mathbf{w}})\hat{\mathbf{w}} - \mathbf{w}T],\ t'' = \gamma[T - \mathbf{r}\cdot\mathbf{w}/c^2], \tag{17}$$

where $\hat{\mathbf{w}} = \mathbf{w}/w$, $\gamma = 1/\sqrt{1-w^2/c^2}$ and $T = \mathbf{r}\cdot\hat{\mathbf{w}}/c$.

The operational derivation of the vector Lorentz transformation validates our operational method

11. OPERATIONAL APPROACH OF THE GROUP PROPERTIES

The main mathematical requirement for a set of coordinate transformations to form a group is that they to accomplish the transitivity property. This stipulates that, successively performed, any two of them engender an equivalent one; *i.e.* both collinear and non-collinear Lorentz transformations form a group. Proving this by the operational method developed in Chs. 7 to 9 requires tracing of radius vectors by light signals. Note that O'_{lf} in Figs. 8 and 9 is the origin of the coordinate system at absolute rest K_A associated to k_A as in Sec. 6 (Sect. 1.1). Tracing $O'_{lf}P_{lB}$ and $O'_{lf}P_C$ in Figs. 8 and 9, respectively, one finds new transformations related to (16) and (17) and similar to them. The light signals will leave O'_{If} when O'_{If} and the origin of k_B in Fig. 8 (that of k'_B in Fig. 9) coincide. They will reach P_{IB} in Fig. 8 (P_{If}, P_C in Fig. 9) simultaneously with the light signal leaving O together with the origins of k_A and k_B, when the origin of k_B reaches O'_{IB} in Fig. 8 ($O'_{IB'}$, O'_{IB} in Fig. 9). As concerns the inverse transformation, it is associated with the motion with constant speed $-v$ of the origin of K from O' to O in Fig. 3 relative to the k now at absolute rest. It connects coordinates and times defining a different event. This because the coordinate system at absolute rest Ξ associated to the moving K by $\xi=\beta^2 x$ differs from that associated with the moving k by $\xi=\beta^2 x'$ [predicted by (25) in view of (24) and (3)].

11.1. For Collinear Lorentz Transformations

Consider the diagram in Fig. 8 for the collinear Lorentz transformations (16), (17). At $t=0$ the coinciding origins of k_A, k_B and a light signal leave the origin O of the coordinate system at absolute rest K. The points O'_A, O'_B in Fig. 8 are reached by the origins of k_A, k_B, respectively, at time T, when the light signal reaches P(X). In accord with Sec. 6 (Sect. 1.1) above, the Lorentz transformations (16), (17) are written at the times βT and γT, respectively. The origin of k_A moves from O'_{IA} to O'_{If} in the time $\gamma T-\beta T$. Analogously to the motion of k relative to K_A in Sec. 6 (Sect. 1.1), we consider the motion of O'_{IB} in relation to O'_{If}. From Fig. 8 we have $\mathbf{r}''=\mathbf{R}-\mathbf{O}'_{If}\mathbf{O}'_{IB}$ with

$$O'_{If}O'_{IB}=w\gamma T-v\gamma T,\ (\mathbf{R}\cdot\hat{\mathbf{w}})=\gamma X-v\gamma T=\gamma(X-vT)=\gamma x'/\beta,$$

where x' is just ξ in (14), and

$$\begin{aligned} x''=(\mathbf{r}''\cdot\hat{\mathbf{w}}) &= \gamma x'/\beta-(w-v)\gamma T=\gamma x'/\beta-\gamma(w-v)\beta t'-\gamma(w-v)\beta v x'/c^2 \\ &=\gamma(x'/\beta)[1-(w-v)v/(c^2-v^2)]-\gamma\beta(w-v)t' \\ &=\gamma\beta(1-wv/c^2)x'-\gamma\beta(w-v)t' \end{aligned}$$

where t' is just τ in (14).

With $\hat{\mathbf{u}}$ given by (4), $\hat{\mathbf{u}}=\mathbf{u}/u$, $\delta=1/\sqrt{1-u^2/c^2}$, and $\hat{\mathbf{v}},\hat{\mathbf{w}},\hat{\mathbf{u}}$ all parallel, the relationships

$$\gamma\beta(1-\mathrm{wv}/\mathrm{c}^2)=\delta,\ \gamma\beta(w-v)=\delta u,\ \text{and } \mathbf{r}'\cdot\hat{\mathbf{u}}=x' \tag{18}$$

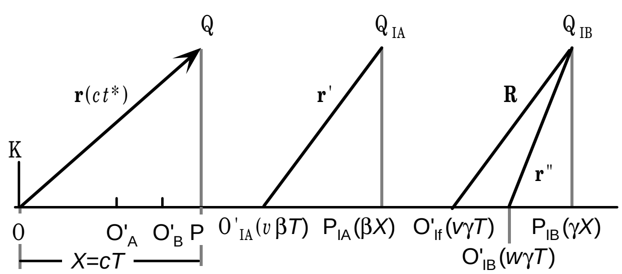

Figure 8.

follow. From the right triangle O'_{IB} Q_{IB} P_{IB} and the right triangle O'_{IA} Q_{IA} P_{IA} (Q_{IA} P_{IA}=Q_{IB} P_{IB}), we get the new vector Lorentz transformation

$$\mathbf{r}''=\mathbf{r}'-(\mathbf{r}'\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}+\delta[(\mathbf{r}'\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}-\mathbf{u}t'],\ \mathrm{t}''=\ \delta\,[\mathrm{t}'-\mathbf{r}'\cdot\mathbf{u}/\mathrm{c}^2],$$

where $t' = \mathbf{r}'\cdot\hat{\mathbf{u}}\,/\,c$ and $t'' = \mathbf{r}''\cdot\hat{\mathbf{u}}\,/\,c$, which relates radius vectors of geometrical points relative to k_B and k_A. Thus the transitivity condition is proved for collinear Lorentz transformations. Therefore, they form a group.

So, together with the operational derivation of the vector Lorentz transformation, the proof that collinear Lorentz transformations form a group validate our operational method.

11.2. For Non-collinear Lorentz transformations

Consider the diagram in Fig. 9. At time $t = 0$ the coordinate systems k_A and k_B, whose origins coincide with that of coordinate system at absolute rest K start moving along non-parallel directions with constant velocities $\mathbf{v}$ and $\mathbf{w}$, respectively. Also at time

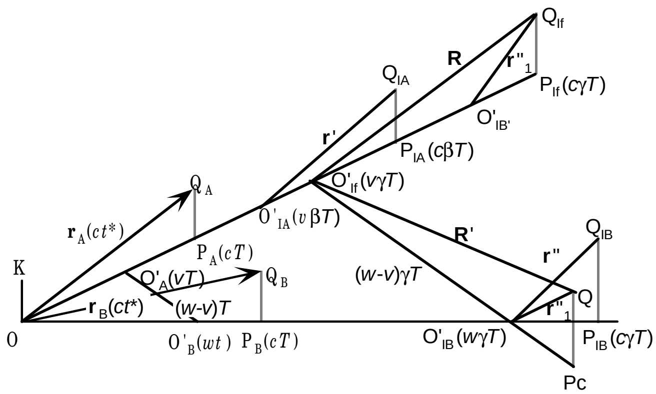

Figure 9.

$t = 0$, light signals start traveling towards P_A and P_B, respectively.

To prove that the resulting non-collinear Lorentz transformations (16), (17) form a group, a light signal and a coordinate system parallel to k_B must move simultaneously at absolute speeds c and $|\mathbf{w} - \mathbf{v}|$ along $O'_AO'_B$ in the time T.

A new Lorentz transformation, in relation with (16) and (17) should follow. To this end, we further consider a coordinate system k'_B parallel to k_B which covers in the time T a distance equal to $OO'_A+O'_AO'_B$ along OP_A at a constant velocity $\mathbf{w}^*$. This coordinate system defines a coordinate system k''_B, also parallel to k_B. The origin of k''_B leaves O'_A at time $t = 0$, and, moving with speed $w^* - v$, reaches O'_B at time T. So we pass from the relative speed $|\mathbf{w} - \mathbf{v}|$ to the relative speed $w^* - v$ by $|\mathbf{w} - \mathbf{v}|\,T = (w^* - v)T$, and from the motion of k_B relative to k_A to one relative to the coordinate system at absolute rest K_A, associated to k_A by $(\mathbf{w}^* - \mathbf{v})T = (T - w^*vT/c^2)\,u\,\hat{\mathbf{u}}$ = with

$$u = (w^* - v)/(1 - w^*v/c^2) \text{ and } \hat{\mathbf{u}} = (\mathbf{w} - \mathbf{v})/|\mathbf{w} - \mathbf{v}|.$$

Using

$$\mathbf{w} - \mathbf{v} = (w^* - v)\hat{\mathbf{u}}, \qquad (19)$$

we have *the operational law of addition of non-parallel speeds.*[2]

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[2] This law has no physical grounding in common with the standard relativistic formula of addition of non-parallel speeds [7] -which predicted the famous, but contested [8] Thomas precession [9]. For the sake of mathematical generality, Thomas missed the physical meaning of the Lorentz transformation by the translation

At the times βT, γT the light signals that leave O simultaneously with k_A, k_B and k'_B reach, respectively, P_{IA} and P_{If}, P_{IB}. The origins of k_A and k_B arrive, respectively, at O'_{IA} and O'_{If}, O'_{IB}. In accord with Sec. 6 (Sect. 1.1), O'_{If} is the origin of the coordinate system at absolute rest K_A at time γT. By the above definition of k'_B and k''_B, the origin of k'_B finds at time γT at a distance equal to $O'_{If}O'_{IB}$ from O'_{If} along OP_{If}, namely at $O'_{IB'}$ in Fig. 9. The light signals leaving O'_{If} simultaneously with the origins of k'_B and k''_B will travel equal distances along the directions of motion of k'_B and k''_B, *viz*. $O'_{If}P_{If}=O'_{If}P_C$. Since $O'_{If}P_{If}$ is the projection of $\mathbf{R}$ onto the direction of $\mathbf{v}$, $O'_{If}P_C$ will be the projection of the radius vector $\mathbf{R'}$ of magnitude R that makes with $\hat{\mathbf{u}}$ an angle equal to that $\mathbf{R}$ makes with $\mathbf{v}$. From $O'_{If}P_{If}=\mathbf{R}\cdot\hat{\mathbf{v}}=\gamma(\mathbf{r}\cdot\hat{\mathbf{v}}-vT)$ and an equation resulting from the first of Eqs. (16), $\mathbf{r'}\cdot\hat{\mathbf{v}}=\beta(\mathbf{r}\cdot\hat{\mathbf{v}}-vT)$, we have $\mathbf{R}\cdot\hat{\mathbf{v}}=(\gamma/\beta)\mathbf{r'}\cdot\hat{\mathbf{v}}$ with

$$(\mathbf{R}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}=(\gamma/\beta)(\mathbf{r'}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}\,. \tag{20}$$

By inserting (20), the inverse of the last of Eqs. (16), and Eq. (19) into $(\mathbf{R}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}-\mathbf{u'}\gamma T$, we obtain:

$$\begin{aligned}
&(\mathbf{R}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}-(\mathbf{w}-\mathbf{v})\gamma T\\
&=\frac{\gamma}{\beta}(\mathbf{r'}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}-\hat{\mathbf{u}}(w^*-v)\gamma\beta t'-\frac{1}{c^2}\hat{\mathbf{u}}(w^*-v)\gamma\beta v(\mathbf{r'}\cdot\hat{\mathbf{v}})\\
&=\frac{\gamma}{\beta}(\mathbf{r'}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}[1-(w^*-v)v/(c^2-v^2)]-\hat{\mathbf{u}}(w^*-v)\gamma\beta t'\\
&=\gamma\beta(1-wv/c^2)(\mathbf{r'}.\hat{\mathbf{v}})\hat{\mathbf{u}}-\hat{\mathbf{u}}(w^*-v)\gamma\beta t'
\end{aligned}$$

In view of Eqs. (20), also valid for w^*, we have:

$$\mathbf{O'}_{IB}\mathbf{P}_C=(\mathbf{R}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}-(\mathbf{w}-\mathbf{v})\gamma T=\delta[(\mathbf{r'}\cdot\hat{\mathbf{v}})-ut']\hat{\mathbf{u}}=\delta[(\mathbf{r'}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}-\mathbf{u}t'].$$

Because $Q_{If}P_{If} = Q_{IB}P_{IB} = QP_C$ by virtue of $Q_AP_A=Q_BP_B$, and $|\mathbf{r}_1''|=|\mathbf{O'}_{IB'}\mathbf{Q}_{If}|=|\mathbf{O'}_{IB}\mathbf{Q}|$ with $\mathbf{O'}_{IB}\mathbf{Q}=\mathbf{QP}_C+\mathbf{O'}_{IB}\mathbf{P}_C$, we have $\mathbf{QP}_C=\mathbf{r'}-(\mathbf{r'}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ and

$$\mathbf{r''}_1=\mathbf{r'}-(\mathbf{r'}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}+\delta\,[(\mathbf{r'}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}-\mathbf{u}t'],\ t''=\delta[t'-\mathbf{r'}\cdot\mathbf{u}/c^2], \tag{21}$$

where $t''=\mathbf{r''}_1\cdot\hat{\mathbf{u}}/c=\mathbf{r''}_1\cdot\hat{\mathbf{w}}/c$. The resulting vector Lorentz transformation (21) proves that the non-collinear Lorentz transformations satisfy the transitivity property. Hence they form a group without requiring rotations of inertial coordinate systems in this aim.

This result validates the Lorentz transformation itself.

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he associated to the vector Lorentz transformation [9]. It was under such condition that the usual matrix multiplication he used to made no physical sense.